Existence of Periodic Solutions for the Discrete-Time Counterpart of a Complex-Valued Hopfield Neural Network with Time-Varying Delays and Impulses

Valéry Covachev
Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
Sofia, Bulgaria
Email: vcovachev@hotmail.com

Zlatinka Covacheva
Middle East College
Muscat, Sultanate of Oman
Email: zkovacheva@hotmail.com

***Abstract*—Using the semi-discretization method, in the present paper a discrete-time counterpart of a complex-valued Hopfield neural network with time-varying delays and impulses is constructed. Sufficient conditions for the existence of periodic solutions of the discrete-time system thus obtained are found in terms of M-matrices by using the continuation theorem of coincidence degree theory.**

I. Introduction

Over the past three decades neural networks have been widely studied since they have been successfully applied to various processing problems such as optimization, image processing, associative memory and many other fields (see [9], [11] and references given therein). Different types of applications depend on the dynamical behaviors of the neural networks.

A. Hirose wrote in the introduction to [12]: "Complex-valued neural networks (CVNNs) are effective and powerful in particular to deal with wave phenomena such as electromagnetic and sonic waves, as well as to process wave-related information ... Researchers extend the world of computation to pattern processing fields based on a novel use of the structure of complex-amplitude (phase and amplitude) information." Further on, he listed the following major application fields of CVNNs: antenna design, estimation of direction of arrival and beamforming of electromagnetic waves, radar imaging, acoustic signal processing and ultrasonic imaging, communications signal processing, image processing, traffic-lights and electric-power systems, quantum devices such as superconductive devices, optical/lightwave information processing including carrier-frequency multiplexing. CVNNs also find applications in fields such as speech synthesis, spatiotemporal analysis of physiological neural devices and systems and artificial neural information processing [23]. CVNNs can be considered as an extension of real-valued neural networks; however, they can be used to solve problems which cannot be solved using their real-valued counterparts, see [19]. The existence, global asymptotic and exponential stability of equilibrium points of CVNNs have been actively studied in the recent years (see [5], [13], [22] and many others). On the other hand, there are very few results on the existence, global asymptotic and exponential stability of periodic solutions of CVNNs [10], [20]. These papers deal with delayed CVNNs, respectively of neutral type and with impulses. In [23], sufficient conditions are obtained for the existence and global asymptotic stability of periodic solutions for delayed complex-valued simplified Cohen-Grossberg neural networks.

In the present paper we construct a discrete-time counterpart of a complex-valued Hopfield network with time-varying delays and impulses by using the semi-discretization method. We find sufficient conditions for the existence of periodic solutions of the discrete-time system thus obtained by using the aforementioned continuation theorem of coincidence degree theory.

II. A continuous-time complex-valued Hopfield neural network and its discrete-time counterpart

We consider the following impulsive neural network with time-varying delays:

$$\begin{aligned}\dot{z}_i(t) &= -a_i(t)z_i(t) + \sum_{j=1}^{m} b_{ij}(t)f_j(z_j(t)) \\ &\quad + \sum_{j=1}^{m} c_{ij}(t)g_j(z_j(t-\tau_{ij}(t))) + I_i(t), \\ &\qquad t > 0, \quad t \neq t_k, \qquad (1)\\ \Delta z_i(t_k) &= -\alpha_{ik}z_i(t_k) + \sum_{j=1}^{m} \beta_{ijk}\Phi_j(z_j(t_k)) \\ &\quad + \sum_{j=1}^{m} \gamma_{ijk}\Gamma_j(z_j(t_k-\tau_{ij}(t_k))) + \zeta_{ik}, \\ &\qquad k \in \{0\} \cup \mathbb{N}, \qquad (2)\\ z_i(s) &= \varphi_i(s), \quad s \in [-\tau, 0], \quad i = \overline{1,m}, \qquad (3)\end{aligned}$$

 978-1-5090-6014-6/18/$31.00

where $z_i(t)$ is the complex-valued state of the i-th neuron at time t; $a_i(t)$ is the rate with which the i-th unit resets its potential to the equilibrium state when isolated from the network and external inputs; $f_j(\cdot)$, $g_j(\cdot)$ denote complex activation functions, respectively without and with delay; the functions $b_{ij}(t)$, $c_{ij}(t)$ represent the weights (or strengths) of the synaptic connections between the j-th neuron and the i-th neuron, respectively without and with transmission delay $\tau_{ij}(t)$; $I_i(t)$ denotes the complex-valued external bias on (input signal introduced from outside the network to) the i-th unit at time t; t_k ($k \in \{0\} \cup \mathbb{N}$) are the moments (instants) of impulse effect satisfying $0 = t_0 < t_1 < t_2 < \cdots < t_k < \cdots$ and $\lim_{k\to\infty} t_k = \infty$; $\Delta z_i(t_k) := z_i(t_k + 0) - z_i(t_k - 0) \equiv z_i(t_k + 0) - z_i(t_k)$ represents the instantaneous change of the state of the i-th neuron at time t_k; $\Phi_j(\cdot)$, $\Gamma_j(\cdot) : \mathbb{C} \to \mathbb{C}$ are some functions; α_{ik}, β_{ijk}, γ_{ijk}, ζ_{ik} are some complex constants; and $\tau = \max_{i,j=\overline{1,m}} \sup_{t>0} \tau_{ij}(t)$.

Here we include a real-life example which is a real-valued neural network of the form (1)–(3) (see, for instance, [1] and [15]):

$$
\begin{aligned}
&C_i\dot{u}_i(t) = -\frac{u_i(t)}{R_i} + \sum_{j=1}^{m} a_{ij} f_j(u_j(t)) \\
&+\sum_{j=1}^{m} b_{ij}(t) g_j(u_j(t-\tau_{ij}(t))) + I_i, \quad t>0,\ t\neq t_k, \\
&\Delta u_i(t_k) = J_{jk}(u_i(t_k)), \qquad k\in\mathbb{N}, \\
&u_i(s) = \varphi_i(s), \quad s\in[-\tau,0], \quad i=\overline{1,m},
\end{aligned}
$$

where $u_i(t)$ denotes the state (voltage) of the i-th neuron at time t, the positive constants C_i and R_i are the neuron amplifier input capacitance and resistance, respectively.

For system (1)–(3) we make the following assumptions:

[H1] There exists a positive number ω and a positive integer p such that

$$
\begin{aligned}
&a_i(t+\omega) = a_i(t),\ I_i(t+\omega) = I_i(t) \ \text{ for} \\
&t \geq 0 \ \text{ and } \ i = \overline{1,m}, \\
&b_{ij}(t+\omega) = b_{ij}(t),\ c_{ij}(t+\omega) = c_{ij}(t), \\
&\tau_{ij}(t+\omega) = \tau_{ij}(t) \text{ for } t\geq 0 \ \text{ and } \ i,j=\overline{1,m}, \\
&t_{k+p} = t_k + \omega \ \text{ for } \ k\in\{0\}\cup\mathbb{N}, \\
&\alpha_{i,k+p} = \alpha_{ik},\ \zeta_{i,k+p} = \zeta_{ik} \ \text{ for} \\
&k\in\{0\}\cup\mathbb{N} \ \text{ and } \ i=\overline{1,m}, \\
&\beta_{ij,k+p} = \beta_{ijk},\ \gamma_{ij,k+p} = \gamma_{ijk} \ \text{ for} \\
&k\in\{0\}\cup\mathbb{N} \ \text{ and } \ i,j=\overline{1,m}.
\end{aligned}
$$

[H2] The complex-valued functions $a_i(t)$, $b_{ij}(t)$, $c_{ij}(t)$ are continuous on $[0,\infty]$; $\mathrm{Re}\, a_i(t) > 0$ for $t \geq 0$ and $0 < \mathrm{Re}\,\alpha_{ik} < 1$ for $k \in \{0\}\cup\mathbb{N}$, $i = \overline{1,m}$.

[H3] There exist positive constants F_j, G_j, $\mathcal{F}_j$, $\mathcal{G}_j$ ($j = \overline{1,m}$) such that

$$
\begin{aligned}
&\max\{|\mathrm{Re}\, f_j(u) - \mathrm{Re}\, f_j(v)|, |\mathrm{Im}\, f_j(u) - \mathrm{Im}\, f_j(v)|\} \\
&\leq\ F_j(|\mathrm{Re}\, u - \mathrm{Re}\, v| + |\mathrm{Im}\, u - \mathrm{Im}\, v|), \\
&\max\{|\mathrm{Re}\, g_j(u) - \mathrm{Re}\, g_j(v)|, |\mathrm{Im}\, g_j(u) - \mathrm{Im}\, g_j(v)|\} \\
&\leq\ G_j(|\mathrm{Re}\, u - \mathrm{Re}\, v| + |\mathrm{Im}\, u - \mathrm{Im}\, v|), \\
&\max\{|\mathrm{Re}\,\Phi_j(u) - \mathrm{Re}\,\Phi_j(v)|, |\mathrm{Im}\,\Phi_j(u) - \mathrm{Im}\,\Phi_j(v)|\} \\
&\leq\ \mathcal{F}_j(|\mathrm{Re}\, u - \mathrm{Re}\, v| + |\mathrm{Im}\, u - \mathrm{Im}\, v|), \\
&\max\{|\mathrm{Re}\,\Gamma_j(u) - \mathrm{Re}\Gamma_j(v)|, |\mathrm{Im}\,\Gamma_j(u) - \mathrm{Im}\Gamma_j(v)|\} \\
&\leq\ \mathcal{G}_j(|\mathrm{Re}\, u - \mathrm{Re}\, v| + |\mathrm{Im}\, u - \mathrm{Im}\, v|)
\end{aligned}
$$

for any $u, v \in \mathbb{C}$.

[H4] The functions $\tau_{ij}(t)$ $(i,j = \overline{1,m})$ are nonnegative and continuous for $t \geq 0$.

[H5] The functions $\varphi_i(s)$ $(i = \overline{1,m})$ are piecewise continuously differentiable on the interval $[-\tau, 0]$, with points of possible discontinuity of the form $t_k - \omega$.

To find an ω-periodic solution of system (1), (2) means to determine the initial functions $\varphi_i(s)$ so that the solution of the initial-value problem (1)–(3) is ω-periodic.

In their paper [14] T. Insperger and G. Stépán presented an efficient numerical method for the stability analysis of linear delayed systems. The semi-discretization method is based on discretization with respect to the past effect only. It was shown that the semi-discretization method is much more effective than the full discretization for the stability analysis. The semi-discretization does not preserve the solutions of the original system. However, it does preserve their exponential stability if the semi-discretization is fine enough in some sense.

A modification of the semi-discretization method was used for the stability analysis of neural networks by S. Mohamad and K. Gopalsamy in [18] and numerous subsequent papers of the same authors. In particular, it can be applied to not necessarily linear neural networks if the nonlinearities satisfy certain conditions.

Similarly to our previous papers [2], [3], henceforth we shall derive a discrete counterpart of system (1)–(3) using a modification of the semi-discretization method and obtain sufficient conditions for the existence of periodic solutions of the latter.

For the sake of definiteness we assume that $\tau \leq \omega$. For a positive integer N we choose the discretization step $h = \omega/N$. For the moment we assume N so large that $h < \min_{k=\overline{1,p}}(t_{k+1} - t_k)$. Then each interval $[nh, (n+1)h]$ contains at most one instant of impulse effect t_k.

For convenience we denote $n = [t/h]$, the greatest integer in t/h, and $n_k = [t_k/h]$. We also denote $\bar{\tau}_{ij}(\cdot) = [\tau_{ij}(\cdot)/h]$, $N_0 = [\tau/h]$.

Let $n \in \{0\}\cup\mathbb{N}$, $n \neq n_k$. This means that the interval $[nh, (n+1)h)$ contains no instant of impulse effect t_k.

We approximate the differential equations (1) on the interval $(nh, (n+1)h)$ by

$$\begin{aligned}&\dot{z}_i(t) + \operatorname{Re} a_i(nh) z_i(t)\\ &= -i\operatorname{Im} a_i(nh) z_i(nh) + I_i(nh) + \sum_{j=1}^{m} b_{ij}(nh) f_j(z_j(nh))\\ &+ \sum_{j=1}^{m} c_{ij}(nh) g_j(z_j((n-\bar{\tau}_{ij}(nh))h)), \quad i=\overline{1,m}.\end{aligned}$$

The factor i before Im is the imaginary unit $\sqrt{-1}$, while everywhere else i is an index, $i = \overline{1,m}$. We multiply both sides of this equation by $e^{\operatorname{Re} a_i(nh)t}$, integrate over the interval $[nh, (n+1)h]$, and then multiply by $e^{-\operatorname{Re} a_i(nh)(n+1)h}$. Thus we obtain

$$\begin{aligned}z_i((n+1)h) - z_i(nh) &= \frac{1-e^{-\operatorname{Re} a_i(nh)h}}{\operatorname{Re} a_i(nh)}\\ \times\Bigg\{ -a_i(nh) z_i(nh) &+ I_i(nh) + \sum_{j=1}^{m} b_{ij}(nh) f_j(z_j(nh))\\ &+ \sum_{j=1}^{m} c_{ij}(nh) g_j(z_j((n-\bar{\tau}_{ij}(nh))h)) \Bigg\}. \end{aligned} \tag{4}$$

Henceforth by abuse of notation we write $z_i(n) := z_i(nh)$ and define $\Delta z_i(n) = z_i(n+1) - z_i(n)$ $(i = \overline{1,m},\ n \in \{0\} \cup \mathbb{N})$. For convenience we adopt the notations:

$$\begin{aligned}A_i(n) &:= \frac{1-e^{-\operatorname{Re} a_i(nh)h}}{\operatorname{Re} a_i(nh)} a_i(nh)\\ &\quad (i=\overline{1,m},\ n \in \mathbb{N}\setminus\{n_k\}_{k\in\mathbb{N}}),\\ I_i(n) &:= \frac{1-e^{-\operatorname{Re} a_i(nh)h}}{\operatorname{Re} a_i(nh)} I_i(nh)\\ &\quad (i=\overline{1,m},\ n \in \{0\}\cup\mathbb{N}\setminus\{n_k\}_{k\in\mathbb{N}}),\\ b_{ij}(n) &:= \frac{1-e^{-\operatorname{Re} a_i(nh)h}}{\operatorname{Re} a_i(nh)} b_{ij}(nh)\\ &\quad (i,j=\overline{1,m},\ n \in \{0\}\cup\mathbb{N}\setminus\{n_k\}_{k\in\mathbb{N}}),\\ c_{ij}(n) &:= \frac{1-e^{-a_i(nh)h}}{a_i(nh)} c_{ij}(nh)\\ &\quad (i,j=\overline{1,m},\ n \in \mathbb{N}\setminus\{n_k\}_{k\in\mathbb{N}}),\\ \tau_{ij}(n) &:= \bar{\tau}_{ij}(nh)\ (i,j=\overline{1,m},\ n\in\{0\}\cup\mathbb{N}).\end{aligned}$$

Clearly, we have $0 < \operatorname{Re} A_i(n) < 1$.

With the above notation equation (4) takes the form

$$\begin{aligned}&\Delta z_i(n) = -A_i(n) z_i(n) + I_i(n)\\ &+\sum_{j=1}^{m} b_{ij}(n) f_j(z_j(n)) + \sum_{j=1}^{m} c_{ij}(n) g_j(z_j(n-\tau_{ij}(n))),\\ &\qquad i=\overline{1,m}, \quad n \neq n_k.\end{aligned} \tag{5}$$

Next, for $n = n_k$ the interval $[nh, (n+1)h]$ contains the instant of impulse effect t_k. On this interval we approximate the impulse condition (2) by

$$\begin{aligned}&\Delta z_i(n_k) = -\alpha_{ik} z_i(n_k) + \zeta_{ik}\\ &+\sum_{j=1}^{m} \beta_{ijk}\Phi_j(z_j(n_k)) + \sum_{j=1}^{m} \gamma_{ijk}\Gamma_j(z_j(n_k - \tau_{ij}(n_k))),\\ &\qquad i=\overline{1,m}, \quad k\in\mathbb{N}.\end{aligned} \tag{6}$$

For uniformity of notation we define

$$A_i(n_k) = \alpha_{ik}, \quad I_i(n_k) = \zeta_{ik} \quad (i=\overline{1,m},\ k\in\{0\}\cup\mathbb{N}).$$

Now the difference system (5), (6) can be written in an operator form as

$$\Delta z = Hz, \tag{7}$$

where

$$\begin{aligned}&(Hz)_i(n) = -A_i(n) z_i(n) + I_i(n)\\ &+\begin{cases}\sum_{j=1}^{m} b_{ij}(n) f_j(z_j(n)) + \sum_{j=1}^{m} c_{ij}(n) g_j(z_j(n-\tau_{ij}(n))), & n \neq n_k,\\ \sum_{j=1}^{m} \beta_{ijk}\Phi_j(z_j(n_k)) + \sum_{j=1}^{m} \gamma_{ijk}\Gamma_j(z_j(n_k-\tau_{ij}(n_k))), & n = n_k.\end{cases}\end{aligned} \tag{8}$$

From the assumptions H1, H2, H4 it follows that

[H6] There exist positive integers N and p such that

$A_i(n+N) = A_i(n),\ I_i(n+N) = I_i(n)$ for $i=\overline{1,m}, \quad n\in\{0\}\cup\mathbb{N}$,

$\tau_{ij}(n+N) = \tau_{ij}(n)$ for $i,j=\overline{1,m},\ n\in\{0\}\cup\mathbb{N}$,

$b_{ij}(n+N) = b_{ij}(n),\ c_{ij}(n+N) = c_{ij}(n)$ for $i,j=\overline{1,m}, \quad n\in\mathbb{N}\setminus\{n_k\}_{k\in\mathbb{N}}$,

$n_{k+p} = n_k + N$ for $k\in\{0\}\cup\mathbb{N}$,

$\beta_{ij,k+p} = \beta_{ijk},\ \gamma_{ij,k+p} = \gamma_{ijk}$ for $k\in\{0\}\cup\mathbb{N}$ and $i,j=\overline{1,m}$.

[H7] $0 < \operatorname{Re} A_i(n) < 1$ for $i=\overline{1,m},\ n \in I_N := \{0, 1, \ldots, N-1\}$, $\tau_{ij}(n) \geq 0$ for $i,j=\overline{1,m},\ n\in I_N$.

We can consider the system (7) for $n\in\{0\}\cup\mathbb{N}$, with initial conditions

$$x_i(\ell) = \varphi_i(\ell) \ \text{ for } \ \ell = 0, -1, \ldots, -N_0,\ i=\overline{1,m}, \tag{9}$$

where $\varphi(\ell) = (\varphi_1(\ell), \varphi_2(\ell), \ldots, \varphi_m(\ell))^T$, $\ell = 0, -1, \ldots, -N_0$, are given initial vectors ($\ell = [s/h],\ \varphi_i(\ell) := \varphi_i(\ell h)$). To find an N-periodic solution of system (7) means to determine the initial vectors $\varphi(\ell)$ so that the solution of the initial-value problem (7), (9) is N-periodic.

III. Main result

In order to formulate our main result we introduce some notation:

For an N-periodic sequence $v(n)$ we denote $\tilde{v} = \sum_{n=0}^{N-1} v(n)$ (if $v(n)$ is given by a longer formula, we write $v\tilde{}$ or $(v)\tilde{}$ instead);

$$\begin{aligned}
\overline{b}_{ij} &= \max\{\sup_{n\neq n_k} |\mathrm{Re}\, b_{ij}(n)|,\ \sup_{n\neq n_k} |\mathrm{Im}\, b_{ij}(n)|\},\\
\overline{c}_{ij} &= \max\{\sup_{n\neq n_k} |\mathrm{Re}\, c_{ij}(n)|,\ \sup_{n\neq n_k} |\mathrm{Im}\, c_{ij}(n)|\},\\
\overline{\beta}_{ij} &= \max\{\max_{k=\overline{1,p}} |\mathrm{Re}\, \beta_{ijk}|,\ \max_{k=\overline{1,p}} |\mathrm{Im}\, \beta_{ijk}|\},\\
\overline{\gamma}_{ij} &= \max\{\max_{k=\overline{1,p}} |\mathrm{Re}\, \gamma_{ijk}|,\ \max_{k=\overline{1,p}} |\mathrm{Im}\, \gamma_{ijk}|\}, \quad i,j=\overline{1,m};
\end{aligned}$$

$$\begin{aligned}
\rho_i = (N-p)\sum_{j=1}^{m} \big[&\overline{b}_{ij}(|\mathrm{Re}\, f_j(0)| + |\mathrm{Im}\, f_j(0)|)\\
&+ \overline{c}_{ij}(|\mathrm{Re}\, g_j(0)| + |\mathrm{Im}\, g_j(0)|)\big]\\
+p \sum_{j=1}^{m} \big[&\overline{\beta}_{ij}(|\mathrm{Re}\, \Phi_j(0)| + |\mathrm{Im}\, \Phi_j(0)|)\\
&+ \overline{\gamma}_{ij}(|\mathrm{Re}\, \Gamma_j(0)| + |\mathrm{Im}\, \Gamma_j(0)|)\big], \quad i=\overline{1,m}; \qquad (10)
\end{aligned}$$

$$\mathcal{B}_{ij} = 2[(N-p)(\overline{b}_{ij}F_j + \overline{c}_{ij}G_j) + p(\overline{\beta}_{ij}\mathcal{F}_j + \overline{\gamma}_{ij}\mathcal{G}_j)], \quad i,j=\overline{1,m}. \qquad (11)$$

Next we introduce the condition

[H8] $$\min_{i=\overline{1,m}} \left(\widetilde{\mathrm{Re}\, A_i} - |\mathrm{Im}\, A_i|\tilde{} - \sum_{j=1}^{m} \mathcal{B}_{ji} \right) > 0.$$

We introduce the $m\times m$ matrices

$$\tilde{\mathcal{A}}_R = \mathrm{diag}\left(\widetilde{\mathrm{Re}\, A_i} \frac{1-\widetilde{\mathrm{Re}\, A_i}}{1+\widetilde{\mathrm{Re}\, A_i}},\ i=\overline{1,m} \right),$$

$$\tilde{\mathcal{A}}_I = \mathrm{diag}\left(|\mathrm{Im}\, A_i|\tilde{},\ i=\overline{1,m}\right), \quad \mathcal{B} = (\mathcal{B}_{ij})_{i,j=1}^{m}, \qquad (12)$$

and the condition

[H9] The $2m\times 2m$ matrix $\mathcal{A} = \begin{pmatrix} \tilde{\mathcal{A}}_R - \mathcal{B} & -\tilde{\mathcal{A}}_I - \mathcal{B} \\ -\tilde{\mathcal{A}}_I - \mathcal{B} & \tilde{\mathcal{A}}_R - \mathcal{B} \end{pmatrix}$ is an M-matrix.

This condition implies that the matrix $\mathcal{A}$ is nonsingular and its inverse has only nonnegative entries [4], [7].

Now we can state our main result as the following theorem.

Theorem 1: Suppose that conditions H3, H6–H9 hold. Then the equation (7) has at least one N-periodic solution.

Proof: We shall prove this theorem using Mawhin's continuation theorem [8, p. 40]. To state this theorem we need some preliminaries (presented as in [3], [16]):

Let $\mathbb{X}, \mathbb{Y}$ be real Banach spaces, $L : \mathrm{Dom}\, L \subset \mathbb{X} \longrightarrow \mathbb{Y}$ be a linear mapping, and $H : \mathbb{X} \longrightarrow \mathbb{Y}$ be a continuous mapping. The mapping L will be called a Fredholm mapping of index zero if $\dim \mathrm{Ker}\, L = \mathrm{codim}\, \mathrm{Im}\, L < +\infty$ and $\mathrm{Im}\, L$ is closed in $\mathbb{Y}$. If L is a Fredholm mapping of index zero and there exist continuous projectors $P : \mathbb{X} \longrightarrow \mathbb{X}$ and $Q : \mathbb{Y} \longrightarrow \mathbb{Y}$ such that $\mathrm{Im}\, P = \mathrm{Ker}\, L$, $\mathrm{Ker}\, Q = \mathrm{Im}\, L = \mathrm{Im}\,(I-Q)$, then the mapping $L|_{\mathrm{Dom}\, L\cap \mathrm{Ker}\, P} : (I-P)\mathbb{X} \longrightarrow \mathrm{Im}\, L$ is invertible. We denote the inverse of this mapping by K_P. If Ω is an open bounded subset of $\mathbb{X}$, the mapping H will be called L-compact on $\overline{\Omega}$ if $QH(\overline{\Omega})$ is bounded and $K_P(I-Q)H : \overline{\Omega} \longrightarrow \mathbb{X}$ is compact. Since $\mathrm{Im}\, Q$ is isomorphic to $\mathrm{Ker}\, L$, there exists an isomorphism $J : \mathrm{Im}\, Q \longrightarrow \mathrm{Ker}\, L$.

Now Mawhin's continuation theorem can be stated as follows.

Lemma 1: Let L be a Fredholm mapping of index zero, let $\Omega \subset \mathbb{X}$ be an open bounded set and let $H : \mathbb{X} \longrightarrow \mathbb{Y}$ be a continuous operator which is L-compact on $\overline{\Omega}$. Assume that the following conditions hold:

(a) for each $\lambda \in (0,1)$, $x \in \partial\Omega \cap \mathrm{Dom}\, L$, $Lx \neq \lambda Hx$;

(b) for each $x \in \partial\Omega \cap \mathrm{Ker}\, L$, $QHx \neq 0$;

(c) $\deg(JQH,\, \Omega\cap \mathrm{Ker}\, L,\, 0) \neq 0$, where $\deg(\cdot,\cdot,\cdot)$ is the Brouwer degree.

Then the equation $Lx = Hx$ has at least one solution in $\overline{\Omega} \cap \mathrm{Dom}\, L$.

Before we proceed further, we shall recall the definition of Brouwer degree [17].

Suppose that M and N are two oriented differentiable manifolds of dimension n (without boundary) with M compact and N connected and suppose that $f : M \to N$ is a differentiable mapping. Let $Df(x)$ denote the differential mapping at the point $x \in M$, that is, the linear mapping $Df(x) : T_x(M) \to T_{f(x)}(N)$. Let $\mathrm{sign}\, Df(x)$ denote the sign of the determinant of $Df(x)$. That is, the sign is positive if f preserves orientation and negative if f reverses orientation.

Definition 1: Let $y \in N$ be a regular value, then we define the *Brouwer degree* (or just *degree*) of f by

$$\deg f \equiv \deg(f, M, y) := \sum_{x\in f^{-1}(y)} \mathrm{sign}\, Df(x).$$

It can be shown that the degree does not depend on the regular value y that we pick so that $\deg f$ is well defined. Note that this degree coincides with the degree as defined for maps of spheres.

In order to apply the theory exposed above to the operator equation (7), we need to express it as an operator equation in a real Banach space. To this end, we denote $x_i = \mathrm{Re}\, z_i$, $y_i = \mathrm{Im}\, z_i$ $(i=\overline{1,m})$, $x = (x_1, x_2, \ldots, x_m)^T$, $y = (y_1, y_2, \ldots, y_m)^T$, and consider $z = (x,y)^T$ as a vector in $\mathbb{R}^{2m}$. Next we redefine the operator H in (7) by the equalities

$$\begin{aligned}
&(Hz)_i(n) = -\mathrm{Re}\, A_i(n)x_i(n) + \mathrm{Im}\, A_i(n)y_i(n) + \mathrm{Re}\, I_i(n)\\
&+\begin{cases}
\displaystyle\sum_{j=1}^{m} [\mathrm{Re}\, b_{ij}(n)\mathrm{Re}\, f_j(z_j(n)) - \mathrm{Im}\, b_{ij}(n)\mathrm{Im}\, f_j(z_j(n)) \\
\quad + \mathrm{Re}\, c_{ij}(n)\mathrm{Re}\, g_j(z_j(n-\tau_{ij}(n))) \\
\quad - \mathrm{Im}\, c_{ij}(n)\mathrm{Im}\, g_j(z_j(n-\tau_{ij}(n)))], \quad n\neq n_k; \\
\displaystyle\sum_{j=1}^{m} [\mathrm{Re}\, \beta_{ijk}\mathrm{Re}\, \Phi_j(z_j(n_k)) - \mathrm{Im}\, \beta_{ijk}\mathrm{Im}\, \Phi_j(z_j(n_k)) \\
\quad + \mathrm{Re}\, \gamma_{ijk}\mathrm{Re}\, \Gamma_j(z_j(n_k-\tau_{ij}(n_k))) \\
\quad - \mathrm{Im}\, \gamma_{ijk}\mathrm{Im}\, \Gamma_j(z_j(n_k-\tau_{ij}(n_k)))], \quad n = n_k,
\end{cases} \qquad (13)
\end{aligned}$$

for $i=\overline{1,m}$, and

$$(Hz)_i(n) = -\mathrm{Re}\, A_{i-m}(n) y_{i-m}(n)$$
$$-\mathrm{Im}\, A_{i-m}(n) x_{i-m}(n) + \mathrm{Im}\, I_{i-m}(n)$$
$$+\begin{cases} \sum\limits_{j=1}^{m} [\mathrm{Re}\, b_{i-m,j}(n) \mathrm{Im}\, f_j(z_j(n)) \\ + \mathrm{Im}\, b_{i-m,j}(n) \mathrm{Re}\, f_j(z_j(n)) \\ + \mathrm{Re}\, c_{i-m,j}(n) \mathrm{Im}\, g_j(z_j(n-\tau_{i-m,j}(n))) \\ + \mathrm{Im}\, c_{i-m,j}(n) \mathrm{Re}\, g_j(z_j(n-\tau_{i-m,j}(n)))]\,,\ n \neq n_k; \\ \sum\limits_{j=1}^{m} [\mathrm{Re}\, \beta_{i-m,jk} \mathrm{Im}\, \Phi_j(z_j(n_k)) \\ + \mathrm{Im}\, \beta_{i-m,jk} \mathrm{Re}\, \Phi_j(z_j(n_k)) \\ + \mathrm{Re}\, \gamma_{i-m,jk} \mathrm{Im}\, \Gamma_j(z_j(n_k-\tau_{i-m,j}(n_k))) \\ + \mathrm{Im}\, \gamma_{i-m,jk} \mathrm{Re}\, \Gamma_j(z_j(n_k-\tau_{i-m,j}(n_k)))]\,,\ n = n_k, \end{cases} \tag{14}$$

for $i=\overline{m+1,2m}$.

Let us choose $\mathbb{X}=\mathbb{Y}=\{z(n)=(x_1(n),x_2(n),\ldots, x_m(n),y_1(n),y_2(m),\ldots,y_m(n))^T : z(n+N)=z(n),\ n\in\{0\}\cup\mathbb{N}\}$. If we define $|x_i|=\max\limits_{n\in I_N}|x_i(n)|$, $|y_i|=\max\limits_{n\in I_N}|y_i(n)|$, $\|z\|=\sum\limits_{i=1}^{m}(|x_i|+|y_i|)$, then $\mathbb{X}$ is a real Banach space with the norm $\|\cdot\|$. For $z\in\mathbb{X}$, let Hz be defined by (13), (14), $Lz=\Delta z$ and

$$Pz = Qz = \frac{1}{N}\tilde{z} = \frac{1}{N}(\tilde{x}_1,\tilde{x}_2,\ldots,\tilde{x}_m,\tilde{y}_1,\tilde{y}_2,\ldots,\tilde{y}_m)^T.$$

Then $\mathrm{Ker}\, L=\{z\in\mathbb{X} : z=c\in\mathbb{R}^{2m}\}$ (vectors with components independent of n), $\mathrm{Im}\, L=\{z\in\mathbb{X} : \sum\limits_{n=0}^{N-1}x_i(n)=\sum\limits_{n=0}^{N-1}y_i(n)=0,\ i=\overline{1,m}\}$ is a closed set in $\mathbb{X}$, and $\mathrm{codim}\,\mathrm{Im}\, L=2m$. Thus L is a Fredholm mapping of index zero. It is easy to see that P and Q are continuous projectors and $\mathrm{Im}\, P=\mathrm{Ker}\, L$, $\mathrm{Im}\, L=\mathrm{Ker}\, Q=\mathrm{Im}\,(I-Q)$, and H is L-compact on $\bar{\Omega}$ for any bounded set $\Omega\subset\mathbb{X}$. Moreover, in condition (c) of Lemma 1 the isomorphism J can be taken as the identity operator I.

Now we will derive some estimates for the solutions z of the operator equation $Lz=\lambda Hz$ for $\lambda\in(0,1)$, that is,

$$\Delta z_i(n) = \lambda (Hz)_i(n), \quad n\in I_N,\ i=\overline{1,2m}. \tag{15}$$

First from (13) and (15), for $n\neq n_k$ we obtain

$$\begin{aligned} &|\Delta x_i(n)| \le \mathrm{Re}\, A_i(n)|x_i(n)| \\ &+ |\mathrm{Im}\, A_i(n)|\,|y_i(n)| + |\mathrm{Re}\, I_i(n)| \\ &+\sum_{j=1}^{m}[|\mathrm{Re}\, b_{ij}(n)||\mathrm{Re}\, f_j(z_j(n))| + |\mathrm{Im}\, b_{ij}(n)||\mathrm{Im}\, f_j(z_j(n))| \\ &+ |\mathrm{Re}\, c_{ij}(n)|\,|\mathrm{Re}\, g_j(z_j(n-\tau_{ij}(n)))| \\ &+ |\mathrm{Im}\, c_{ij}(n)|\,|\mathrm{Im}\, g_j(z_j(n-\tau_{ij}(n)))|] \\ &\le \mathrm{Re}\, A_i(n)|x_i(n)| + |\mathrm{Im}\, A_i(n)|\,|y_i(n)| + |\mathrm{Re}\, I_i(n)| \\ &+\sum_{j=1}^{m}\overline{b}_{ij}\left[2F_j(|x_j(n)|+|y_j(n)|) + |\mathrm{Re}\, f_j(0)| + |\mathrm{Im}\, f_j(0)|\right] \\ &+\sum_{j=1}^{m}\overline{c}_{ij}\left[2G_j(|x_j(n-\tau_{ij}(n))| + |x_j(n-\tau_{ij}(n))|)\right. \\ &\left.+ |\mathrm{Re}\, g_j(0)| + |\mathrm{Im}\, g_j(0)|\right] \\ &\le \mathrm{Re}\, A_i(n)|x_i(n)| + |\mathrm{Im}\, A_i(n)|\,|y_i(n)| + |\mathrm{Re}\, I_i(n)| \\ &+\sum_{j=1}^{m}2\left(\overline{b}_{ij}F_j+\overline{c}_{ij}G_j\right)(|x_j|+|y_j|) \\ &+\sum_{j=1}^{m}\left[\overline{b}_{ij}(|\mathrm{Re}\, f_j(0)|+|\mathrm{Im}\, f_j(0)|)\right. \\ &\left.+ \overline{c}_{ij}(|\mathrm{Re}\, g_j(0)|+|\mathrm{Im}\, g_j(0)|)\right]. \end{aligned}$$

Similarly, for $n=n_k$ we have

$$\begin{aligned} &|\Delta x_i(n_k)| \le \mathrm{Re}\, A_i(n_k)|x_i| + |\mathrm{Im}\, A_i(n_k)|\,|y_i| \\ &+|\mathrm{Re}\, I_i(n_k)| + \sum_{j=1}^{m}2\left(\overline{\beta}_{ij}\mathcal{F}_j+\overline{\gamma}_{ij}\mathcal{G}_j\right)(|x_j|+|y_j|) \\ &+\sum_{j=1}^{m}\left[\overline{\beta}_{ij}(|\mathrm{Re}\,\Phi_j(0)|+|\mathrm{Im}\,\Phi_j(0)|)\right. \\ &\left.+ \overline{\gamma}_{ij}(|\mathrm{Re}\,\Gamma_j(0)|+|\mathrm{Im}\,\Gamma_j(0)|)\right]. \end{aligned}$$

From the above inequalities we obtain

$$\begin{aligned} &\sum_{n=0}^{N-1}|\Delta x_i(n)| \le \widetilde{\mathrm{Re}\, A_i}|x_i| + |\mathrm{Im}\, A_i|^{\sim}|y_i| + |\mathrm{Re}\, I_i|^{\sim} \\ &+(N-p)\sum_{j=1}^{m}\left[\overline{b}_{ij}(|\mathrm{Re}\, f_j(0)|+|\mathrm{Im}\, f_j(0)|)\right. \\ &\left.+ \overline{c}_{ij}(|\mathrm{Re}\, g_j(0)|+|\mathrm{Im}\, g_j(0)|)\right] \\ &+p\sum_{j=1}^{m}\left[\overline{\beta}_{ij}(|\mathrm{Re}\,\Phi_j(0)|+|\mathrm{Im}\,\Phi_j(0)|)\right. \\ &\left.+ \overline{\gamma}_{ij}(|\mathrm{Re}\,\Gamma_j(0)|+|\mathrm{Im}\,\Gamma_j(0)|)\right] \\ &+2\sum_{j=1}^{m}\left[(N-p)\left(\overline{b}_{ij}F_j+\overline{c}_{ij}G_j\right)\right. \\ &\left.+p\left(\overline{\beta}_{ij}\mathcal{F}_j+\overline{\gamma}_{ij}\mathcal{G}_j\right)\right](|x_j|+|y_j|) \end{aligned}$$

or, using the notations (10) and (11),

$$\begin{aligned} \sum_{n=0}^{N-1}|\Delta x_i(n)| &\le \widetilde{\mathrm{Re}\, A_i}|x_i| + |\mathrm{Im}\, A_i|^{\sim}|y_i| + \rho_i + |\mathrm{Re}\, I_i|^{\sim} \\ &+ \sum_{j=1}^{m}\mathcal{B}_{ij}(|x_j|+|y_j|). \end{aligned} \tag{16}$$

Similarly, we obtain

$$\begin{aligned} \sum_{n=0}^{N-1}|\Delta y_i(n)| &\le \widetilde{\mathrm{Re}\, A_i}|y_i| + |\mathrm{Im}\, A_i|^{\sim}|x_i| + \rho_i + |\mathrm{Im}\, I_i|^{\sim} \\ &+ \sum_{j=1}^{m}\mathcal{B}_{ij}(|x_j|+|y_j|). \end{aligned} \tag{17}$$

For $i=\overline{1,m}$, summing up the i-th equation of (15) over $n\in I_N$, we obtain

$$\sum_{n=0}^{N-1}\operatorname{Re}A_i(n)x_i(n)=\sum_{n=0}^{N-1}\operatorname{Im}A_i(n)y_i(n)+\sum_{n=0}^{N-1}\operatorname{Re}I_i(n)$$
$$+\sum_{j=1}^{m}\Bigg\{\sum_{n\in\Im_N}[\operatorname{Re}b_{ij}(n)\operatorname{Re}f_j(z_j(n))-\operatorname{Im}b_{ij}(n)\operatorname{Im}f_j(z_j(n))$$
$$+\operatorname{Re}c_{ij}(n)\operatorname{Re}g_j(z_j(n-\tau_{ij}(n)))$$
$$-\operatorname{Im}c_{ij}(n)\operatorname{Im}g_j(z_j(n-\tau_{ij}(n)))]$$
$$+\sum_{k=1}^{p}[\operatorname{Re}\beta_{ijk}\operatorname{Re}\Phi_j(z_j(n_k))-\operatorname{Im}\beta_{ijk}\operatorname{Im}\Phi_j(z_j(n_k))$$
$$+\operatorname{Re}\gamma_{ijk}\operatorname{Re}\Gamma_j(z_j(n_k-\tau_{ij}(n_k)))$$
$$-\operatorname{Im}\gamma_{ijk}\operatorname{Im}\Gamma_j(z_j(n_k-\tau_{ij}(n_k)))]\Bigg\},$$

where $\Im_N=I_N\setminus\{n_k\}_{k=0}^{p-1}$. As above we obtain

$$\left|\sum_{n=0}^{N-1}\operatorname{Re}A_i(n)x_i(n)\right|$$
$$\le|\operatorname{Im}A_i|\tilde{}\,|y_i|+\rho_i+|\operatorname{Re}I_i|\tilde{}+\sum_{j=1}^{m}\mathcal{B}_{ij}(|x_j|+|y_j|).\quad(18)$$

In a similar way, we find

$$\left|\sum_{n=0}^{N-1}\operatorname{Re}A_i(n)y_i(n)\right|$$
$$\le|\operatorname{Im}A_i|\tilde{}\,|x_i|+\rho_i+|\operatorname{Im}I_i|\tilde{}+\sum_{j=1}^{m}\mathcal{B}_{ij}(|x_j|+|y_j|).\quad(19)$$

Now we shall use the following lemma (see [6], [21]).

Lemma 2: Let $v:\ \mathbb{Z}\to\mathbb{R}$ be N-periodic, i.e., $v(n+N)=v(n)$ for any $n\in\mathbb{Z}$. Then for any fixed $\nu_1,\nu_2\in I_N$ and any $n\in\mathbb{Z}$ we have

$$v(\nu_2)-\sum_{k=0}^{N-1}|v(k+1)-v(k)|$$
$$\le v(n)\le v(\nu_1)+\sum_{k=0}^{N-1}|v(k+1)-v(k)|.$$

According to Lemma 2, for arbitrary $n,\nu_1,\nu_2\in I_N$ we have

$$x_i(\nu_2)-\sum_{n=0}^{N-1}|\Delta x_i(n)|$$
$$\le x_i(n)\le x_i(\nu_1)+\sum_{n=0}^{N-1}|\Delta x_i(n)|.$$

We multiply these inequalities by $\operatorname{Re}A_i(n)$ and sum up over I_N to obtain

$$\widetilde{\operatorname{Re}A_i}x_i(\nu_2)-\widetilde{\operatorname{Re}A_i}\sum_{n=0}^{N-1}|\Delta x_i(n)|$$
$$\le\sum_{n=0}^{N-1}\operatorname{Re}A_i(n)x_i(n)\le\widetilde{\operatorname{Re}A_i}x_i(\nu_1)+\widetilde{\operatorname{Re}A_i}\sum_{n=0}^{N-1}|\Delta x_i(n)|.$$

Let $|x_i(\nu_0)|=|x_i|\equiv\max_{n\in I_N}|x_i(n)|$. If $x_i(\nu_0)\ge0$, we choose $\nu_2=\nu_0$. Then

$$\widetilde{\operatorname{Re}A_i}|x_i|=\widetilde{\operatorname{Re}A_i}x_i(\nu_2)$$
$$\le\sum_{n=0}^{N-1}\operatorname{Re}A_i(n)x_i(n)+\widetilde{\operatorname{Re}A_i}\sum_{n=0}^{N-1}|\Delta x_i(n)|.$$

If $x_i(\nu_0)<0$, we choose $\nu_1=\nu_0$,

$$\widetilde{\operatorname{Re}A_i}|x_i|=-\widetilde{\operatorname{Re}A_i}x_i(\nu_1)$$
$$\le-\sum_{n=0}^{N-1}\operatorname{Re}A_i(n)x_i(n)+\widetilde{\operatorname{Re}A_i}\sum_{n=0}^{N-1}|\Delta x_i(n)|.$$

Thus in both cases we have

$$\widetilde{\operatorname{Re}A_i}|x_i|\le\left|\sum_{n=0}^{N-1}\operatorname{Re}A_i(n)x_i(n)\right|+\widetilde{\operatorname{Re}A_i}\sum_{n=0}^{N-1}|\Delta x_i(n)|.$$

Making use of the estimates (16) and (18), we obtain

$$\widetilde{\operatorname{Re}A_i}|x_i|\le|\operatorname{Im}A_i|\tilde{}\,|y_i|+\rho_i+|\operatorname{Re}I_i|\tilde{}$$
$$+\sum_{j=1}^{m}\mathcal{B}_{ij}(|x_j|+|y_j|)+\widetilde{\operatorname{Re}A_i}\Bigg[\widetilde{\operatorname{Re}A_i}|x_i|+|\operatorname{Im}A_i|\tilde{}\,|y_i|$$
$$+\rho_i+|\operatorname{Re}I_i|\tilde{}+\sum_{j=1}^{m}\mathcal{B}_{ij}(|x_j|+|y_j|)\Bigg]=(\widetilde{\operatorname{Re}A_i})^2|x_i|$$
$$+(1+\widetilde{\operatorname{Re}A_i})\Big[|\operatorname{Im}A_i|\tilde{}\,|y_i|+\rho_i$$
$$+|\operatorname{Re}I_i|\tilde{}+\sum_{j=1}^{m}\mathcal{B}_{ij}(|x_j|+|y_j|\Big]$$

or

$$\widetilde{\operatorname{Re}A_i}\frac{1-\widetilde{\operatorname{Re}A_i}}{1+\widetilde{\operatorname{Re}A_i}}|x_i|-|\operatorname{Im}A_i|\tilde{}\,|y_i|-\sum_{j=1}^{m}\mathcal{B}_{ij}(|x_j|+|y_j|)$$
$$\le\rho_i+|\operatorname{Re}I_i|\tilde{}.\quad(20)$$

Similarly,

$$\widetilde{\operatorname{Re}A_i}\frac{1-\widetilde{\operatorname{Re}A_i}}{1+\widetilde{\operatorname{Re}A_i}}|y_i|-|\operatorname{Im}A_i|\tilde{}\,|x_i|-\sum_{j=1}^{m}\mathcal{B}_{ij}(|x_j|+|y_j|)$$
$$\le\rho_i+|\operatorname{Im}I_i|\tilde{}.\quad(21)$$

If we introduce the m-dimensional vectors

$$|\boldsymbol{x}|=(|x_1|,\dots,|x_m|)^T,\quad|\boldsymbol{y}|=(|y_1|,\dots,|y_m|)^T,$$
$$\boldsymbol{\rho}=(\rho_1,\dots,\rho_m)^T,\ \widetilde{|\operatorname{Re}I|}=(|\operatorname{Re}I_1|\tilde{},\dots,|\operatorname{Re}I_m|\tilde{})^T$$

and $\widetilde{|\operatorname{Im}I|}=(|\operatorname{Im}I_i|\tilde{},\dots,|\operatorname{Im}I_i|\tilde{})^T$, then the system of inequalities (20), (21) for $i=\overline{1,m}$ can be written in a matrix form

$$(\tilde{\mathcal{A}}_R-\mathcal{B})|\boldsymbol{x}|-(\tilde{\mathcal{A}}_I+\mathcal{B})|\boldsymbol{y}|=\boldsymbol{\rho}+\widetilde{|\operatorname{Re}I|},$$
$$-(\tilde{\mathcal{A}}_I+\mathcal{B})|\boldsymbol{x}|+(\tilde{\mathcal{A}}_R-\mathcal{B})|\boldsymbol{y}|=\boldsymbol{\rho}+\widetilde{|\operatorname{Im}I|},$$

where the matrices $\tilde{\mathcal{A}}_R$, $\tilde{\mathcal{A}}_I$ and $\mathcal{B}$ were introduced in (12). We further introduce the $2m$-dimensional vectors $|\boldsymbol{z}|=$

$(|\boldsymbol{x}|, |\boldsymbol{y}|)^T$, $\boldsymbol{\rho}' = (\boldsymbol{\rho}, \boldsymbol{\rho})^T$ and $\tilde{I} = (\widetilde{|\mathrm{Re}\, I|}, \widetilde{|\mathrm{Im}\, I|})^T$, then the above matrix system can be written in the form

$$\mathcal{A}|\boldsymbol{z}| \leq \boldsymbol{\rho}' + \tilde{I}. \tag{22}$$

By virtue of condition H9, the inequality (22) implies

$$|\boldsymbol{z}| \leq \mathcal{A}^{-1}(\boldsymbol{\rho}' + \tilde{I}).$$

If $\mathcal{A}^{-1}(\boldsymbol{\rho}' + \tilde{I}) = (C_1^*, C_2^*, \ldots, C_{2m}^*)^T$, this means that the components of each solution of $\Delta z = \lambda H z$ satisfy $|x_i| \leq C_i^*$, $|y_i| \leq C_{i+m}^*$. If we denote $C^* = \sum_{i=1}^{2m} C_i^*$, then each solution of $\Delta z = \lambda H z$ satisfies $\|z\| \leq C^*$. Now we take $\Omega = \{z \in \mathbb{X} : \|z\| < C\}$, where $C > C^*$ will be chosen later. Obviously, Ω satisfies condition (a) of Lemma 1.

Next, let $z \in \partial\Omega \cap \mathrm{Ker}\, L = \partial\Omega \cap \mathbb{R}^{2m}$, i.e., z is a constant vector in $\mathbb{R}^{2m}$ with $\|z\| = C$. For such z: if $i = \overline{1, m}$, then

$$\begin{aligned}
N(QHz)_i &= -\widetilde{\mathrm{Re}\, A_i} x_i + \widetilde{\mathrm{Im}\, A_i} y_i + \widetilde{\mathrm{Re}\, I_i} \\
&+ \sum_{j=1}^{m} \Bigg\{ \sum_{n \in \Im_n} [(\mathrm{Re}\, b_{ij}(n) \mathrm{Re}\, f_j(z_j) - \mathrm{Im}\, b_{ij}(n) \mathrm{Im}\, f_j(z_j)) \\
&\quad + (\mathrm{Re}\, c_{ij}(n) \mathrm{Re}\, g_j(z_j) - \mathrm{Im}\, c_{ij}(n) \mathrm{Im}\, g_j(z_j))] \\
&+ \sum_{k=1}^{p} [(\mathrm{Re}\, \beta_{ijk} \mathrm{Re}\, \Phi_j(z_j) - \mathrm{Im}\, \beta_{ijk} \mathrm{Im}\, \Phi_j(z_j)) \\
&\quad + (\mathrm{Re}\, \gamma_{ijk} \mathrm{Re}\, \Gamma_j(z_j) - \mathrm{Im}\, \gamma_{ijk} \mathrm{Im}\, \Gamma_j(z_j))] \Bigg\}
\end{aligned}$$

and

$$\begin{aligned}
N|(QHz)_i| &\geq \widetilde{\mathrm{Re}\, A_i} |x_i| - |\mathrm{Im}\, A_i|\tilde{}\, |y_i| - |\widetilde{\mathrm{Re}\, I_i}| \\
&- 2 \sum_{j=1}^{m} \left\{ (N-p) \left[\overline{b}_{ij} F_j + \overline{c}_{ij} G_j \right] + p \left[\overline{\beta}_{ij} \mathcal{F}_j + \overline{\gamma}_{ij} \mathcal{G}_j \right] \right\} \\
&\qquad \times (|x_j| + |y_j|) \\
&- \sum_{j=1}^{m} \left\{ (N-p) \left[\overline{b}_{ij} (|\mathrm{Re}\, f_j(0)| + |\mathrm{Im}\, f_j(0)|) \right. \right. \\
&\quad \left. + \overline{c}_{ij} (|\mathrm{Re}\, g_j(0)| + |\mathrm{Im}\, g_j(0)|) \right] \\
&\quad + p \left[\overline{\beta}_{ij} (|\mathrm{Re}\, \Phi_j(0)| + |\mathrm{Im}\, \Phi_j(0)|) \right. \\
&\quad \left. \left. + \overline{\gamma}_{ij} (|\mathrm{Re}\, \Gamma_j(0)| + |\mathrm{Im}\, \Gamma_j(0)|) \right] \right\} \\
&= \widetilde{\mathrm{Re}\, A_i} |x_i| - |\mathrm{Im}\, A_i|\tilde{}\, |y_i| - \sum_{j=1}^{m} \mathcal{B}_{ij} (|x_j| + |y_j|) \\
&\qquad - \rho_i - |\widetilde{\mathrm{Re}\, I_i}|
\end{aligned}$$

and, similarly, if $i = \overline{m+1, 2m}$, then

$$\begin{aligned}
N|(QHz)_i| &\geq (\mathrm{Re}\, A_{i-m})\tilde{}\, |y_{i-m}| - |\mathrm{Im}\, A_{i-m}|\tilde{}\, |x_{i-m}| \\
&- \sum_{j=1}^{m} \mathcal{B}_{i-m,j} (|x_j| + |y_j|) - \rho_{i-m} - |(\mathrm{Im}\, I_{i-m})\tilde{}\, |.
\end{aligned}$$

Thus,

$$\begin{aligned}
N\|QHz\| &= \sum_{i=1}^{m} N|(QHz)_i| \\
&\geq \sum_{i=1}^{m} \left(\widetilde{\mathrm{Re}\, A_i} - |\mathrm{Im}\, A_i|\tilde{} - \sum_{j=1}^{m} \mathcal{B}_{ji} \right) (|x_i| + |y_i|) \\
&\quad - \sum_{i=1}^{m} \left(2\rho_i + |\widetilde{\mathrm{Re}\, I_i}| + |(\mathrm{Im}\, I_{i-m})\tilde{}\, | \right) \\
&\geq \min_{i=\overline{1,m}} \left(\widetilde{\mathrm{Re}\, A_i} - |\mathrm{Im}\, A_i|\tilde{} - \sum_{j=1}^{m} \mathcal{B}_{ji} \right) \|z\| - \|\boldsymbol{\rho}'\| - \|\tilde{I}\| \\
&= \min_{i=\overline{1,m}} \left(\widetilde{\mathrm{Re}\, A_i} - |\mathrm{Im}\, A_i|\tilde{} - \sum_{j=1}^{m} \mathcal{B}_{ji} \right) C - \|\boldsymbol{\rho}'\| - \|\tilde{I}\|.
\end{aligned}$$

By condition H8,

$$\min_{i=\overline{1,m}} \left(\widetilde{\mathrm{Re}\, A_i} - |\mathrm{Im}\, A_i|\tilde{} - \sum_{j=1}^{m} \mathcal{B}_{ji} \right) > 0.$$

Then we can choose $C > C^*$ so large that

$$\min_{i=\overline{1,m}} \left(\widetilde{\mathrm{Re}\, A_i} - |\mathrm{Im}\, A_i|\tilde{} - \sum_{j=1}^{m} \mathcal{B}_{ji} \right) C > \|\boldsymbol{\rho}'\| + \|\tilde{I}\|.$$

Hence for $z \in \partial\Omega \cap \mathrm{Ker}\, L$ we have $N\|QHz\| > 0$ and $QHz \neq 0$, that is, condition (b) of Lemma 1 is satisfied.

To prove (c), we define the mapping

$$(QH)_\mu : \mathrm{Dom}\, L \times [0, 1] \longrightarrow \mathbb{X}$$

by

$$(QH)_\mu = -\frac{\mu}{N} \widetilde{\mathrm{Re}\, A} + (1 - \mu) QH,$$

where

$$\widetilde{\mathrm{Re}\, A} z = (\widetilde{\mathrm{Re}\, A_1} x_1, \ldots, \widetilde{\mathrm{Re}\, A_m} x_m, \widetilde{\mathrm{Re}\, A_1} y_1, \ldots, \widetilde{\mathrm{Re}\, A_m} y_m)^T.$$

As above, for $z \in \partial\Omega \cap \mathrm{Ker}\, L$ we obtain

$$\begin{aligned}
\|(QH)_\mu z\| \geq \frac{1}{N} \Bigg\{ &\min_{i=\overline{1,m}} \left(\widetilde{\mathrm{Re}\, A_i} - |\mathrm{Im}\, A_i|\tilde{} - \sum_{j=1}^{m} \mathcal{B}_{ji} \right) C \\
&- \|\boldsymbol{\rho}'\| - \|\tilde{I}\| \Bigg\} > 0.
\end{aligned}$$

This means that $(QH)_\mu z \neq 0$ for $z \in \partial\Omega \cap \mathrm{Ker}\, L$ and $\mu \in [0, 1]$. From the homotopy invariance of the Brouwer degree, it follows that

$$\begin{aligned}
&\deg(QH, \Omega \cap \mathrm{Ker}\, L, 0) \\
&= \deg\left(-\frac{1}{N} \widetilde{\mathrm{Re}\, A}, \Omega \cap \mathrm{Ker}\, L, 0 \right) = (-1)^{2m} \neq 0.
\end{aligned}$$

According to Lemma 1 equation (7) has at least one N-periodic solution. This completes the proof of Theorem 1. ∎

IV. Conclusion

In the present paper, by using the semi-discretization method we constructed a discrete-time counterpart of a complex-valued Hopfield neural network with time-varying delays and impulses. We found sufficient conditions for the existence of a periodic solution of the discrete-time system thus obtained in terms of M-matrices by using the continuation theorem of coincidence degree theory. The uniqueness of the periodic solution will be studied in a forthcoming paper.

References

[1] H. Akça, R. Alassar and V. Covachev, "Stability of neural networks with time varying delays in the presence of impulses," *Adv. Dyn. Syst. Appl.*, vol. 1, no. 1, pp. 1–15, 2006.

[2] H. Akça, E. Al-Zahrani, V. Covachev and Z. Covacheva, "Existence of periodic solutions for the discrete-time counterpart of a neutral-type cellular neural network with time-varying delays and impulses," *Int. J. Appl. Math. Stat.*, vol. 57 , no. 1, pp. 154–166, 2018.

[3] H. Akça, V. Covachev, Z. Covacheva and S. Mohamad, "Global exponential periodicity for the discrete analogue of an impulsive Hopfield neural network with finite distributed delays," *Funct. Differ. Equ.*, vol. 16, no. 1, pp. 53–72, 2009.

[4] A. Berman and R. J. Plemmons, *Nonnegative Matrices in Mathematical Sciences*, New York: Academic Press, 1979.

[5] M. Bohner, S. H. Rao and S. Sanyal, "Global stability of complex-valued neural networks on time scales," *Differ. Equ. Dyn. Syst.*, vol. 19, no. 1–2, pp. 3–11, 2011.

[6] M. Fan and K. Wang, "Periodic solutions of a discrete time nonautonomous ratio-dependent predator-prey system," *Math. Comput. Model.*, vol. 35, no. 9–10, pp. 951–961, 2002.

[7] M. Fiedler, *Special Matrices and Their Applications in Numerical Mathematics*, Dordrecht: Martinus Nijhoff, 1986.

[8] R. E. Gaines and J. L. Mawhin, *Coincidence Degree and Nonlinear Differential Equations*, Berlin-Heidelberg: Springer-Verlag, 1977.

[9] A. I. Galushkin, *Neural Networks Theory*, Berlin-Heidelberg: Springer-Verlag, 2007.

[10] S. Guo and B. Du, "Global exponential stability of periodic solution for neutral-type complex-valued recurrent neural networks," *Discrete Dyn. Nat. Soc.*, vol. 2016, Article ID 1267954, 10 pp., 2016.

[11] J. Heaton, *Introduction to the Math of Neural Networks*, Heaton Research, ISBN: 9781604390339, 2011.

[12] A. Hirose (Ed.), *Complex-Valued Neural Networks: Advances and Applications*, Wiley-IEEE Press, 2013.

[13] J. Hu and J. Wang, "Global stability of complex-valued recurrent neural networks with time-delays," *IEEE Trans. Neural Netw. Learn. Syst.*, vol. 23, no. 6, pp. 853–865, 2012.

[14] T. Insperger and G. Stépán, "Semi-discretization method for delayed systems," *Int. J. Numer. Math. Engng.*, vol. 55, pp. 503–518, 2002.

[15] Y. Li and M. Hua, "The stability analysis for a kind of impulsive Hopfield cellular neural networks," *3rd International Conference on Mechatronics, Robotics and Automation (ICMRA 2015)*, Atlantic Press, 4 p., 2015.

[16] Y. Li, L. Zhao and X. Chen, "Existence of periodic solutions for neural type cellular neural networks with delays," Appl. Math. Model., vol. 36, no. 3, pp. 1173–1183, 2012.

[17] J. W. Milnor, *Topology from the Differentiable Viewpoint*, Charlottesville: The University Press of Virginia, 1969.

[18] S. Mohamad and K. Gopalsamy, "Dynamics of a class of discrete-time neural networks and their continuous-time counterparts," *Math. Comput. Simulation*, vol. 53, pp. 1–39, 2000.

[19] T. Nitta, "Solving the XOR problem and the detections of symmetry using a single complex-valued neuron," *Neural Netw.*, vol. 16, no. 8, pp. 1101–1105, 2003.

[20] D. Xie and Y. P. Jiang, "Global exponential stability of periodic solutions for delayed complex-valued neural networks with impulses," *Neurocomputing*, vol. 207, pp. 528–538, 2016.

[21] R. Xu, L. Chen and F. Hao, "Periodic solutions of discrete-time Lotka-Volterra type food-chain model with delays," *Appl. Math. Comput.*, vol. 171, no. 1, pp. 91–103, 2005.

[22] Y. Zhang and C. Lin, "Global stability criterion for delayed complex-valued recurrent neural networks," *IEEE Trans. Neural Netw. Learn. Syst.*, vol. 25, no. 9, pp. 1704–1708, 2014.

[23] Z. Zhang and T. Zheng, "Global asymptotic stability of periodic solutions for delayed complex-valued Cohen-Grossberg neural networks by combining coincidence degree theory with LMI method," *Neurocomputing*, vol. 289, pp. 220–230, 2018.